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Fixed Point Theorems in Fuzzy Probability Metric Spaces: Existence and Uniqueness Under Fuzziness

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Abstract— ***This paper extends fixed point theory to fuzzy probability metric spaces (FPMs) and investigates how fuzziness impacts the existence and uniqueness of fixed points. By integrating fuzzy set theory with probability metric spaces, we offer theoretical insights and practical applications across diverse fields. The development of new theorems and methodologies specific to FPMs in this study provides a robust and reliable framework for addressing uncertainties inherent in various mathematical and real-world problems, instilling confidence in the applicability of our research.***

Index Terms— ***Fixed Point Theorems, Fuzzy Probability Metric Spaces (FPMs), Fuzziness, Existence and Uniqueness, Fuzzy Set Theory, Probability Metric Spaces, Iterative Methods, Multi-Valued Mappings***

I. INTRODUCTION

Background and Motivation

Fixed point theorems, such as those by Banach, Brouwer, and Schauder, are fundamental in mathematical analysis. They provide crucial tools for proving the existence and uniqueness of solutions to various equations, which are essential in diverse fields, including differential equations, optimisation, and dynamic systems. These theorems have been extensively studied and applied in classical metric spaces, offering profound insights and solutions to complex mathematical problems.

Probability metric spaces ($\mathcal{P\&MS}$) extend the concept of metric spaces to encompass probability distributions, enabling the measurement of distances between random variables or distributions. These spaces are particularly useful in stochastic processes, where they facilitate the analysis and modelling of random phenomena. Applications of probability metric spaces range from statistical mechanics and information theory to economics and finance, where uncertainty and randomness play pivotal roles.

Lotfi Zadeh introduced fuzzy set theory in the 1960s. It addresses real-world data's inherent uncertainties and imprecisions. Unlike classical set theory, which requires precise membership, fuzzy set theory allows for degrees of membership, making it ideal for modelling ambiguous or vague information. This flexibility is precious in control systems, decision-making, and artificial intelligence fields.

Combining these concepts and extending fixed point theorems to fuzzy probability metric spaces is a natural and necessary advancement. This integration allows for analysing systems with probabilistic and fuzzy uncertainties, broadening the scope of fixed-point theory. Such an extension enhances theoretical understanding and provides powerful tools for solving practical problems involving complex uncertainties, contributing to advancements in various scientific and engineering disciplines.

Objectives:

- Formulate and prove fixed point theorems within fuzzy probability metric spaces.
- Analyse the conditions under which fuzziness affects the existence and uniqueness of fixed points.
- Explore applications of these extended theorems in fields such as stochastic processes, fuzzy logic, and uncertainty modelling.

II. LITERATURE REVIEW

Fixed Point Theorems

Classical fixed-point theorems form the backbone of fixed-point theory. Banach's Fixed Point Theorem, also known as the Contraction Mapping Theorem, is pivotal for proving the existence and uniqueness of fixed points in complete metric spaces. Brouwer's Fixed Point Theorem asserts the existence of fixed points for continuous mappings on compact convex sets in Euclidean spaces, with significant implications in fields such as economics and game theory. Schauder's Fixed Point Theorem extends Brouwer's theorem to infinite-dimensional spaces, broadening its applications to functional analysis and partial differential equations.

Extensions of these classical theorems to probabilistic and metric spaces have been explored extensively. For instance, probabilistic extensions of Banach's theorem cater to stochastic processes, where mappings involve random variables and their distributions. Similarly, extensions in metric spaces have addressed mappings in more generalised contexts, including those with non-linear or multi-valued functions.

Fuzzy Set Theory

Fuzzy set theory, introduced by Zadeh in 1965, allows for degrees of membership, providing a flexible framework for

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modelling uncertainty. Fuzzy numbers and their arithmetic extend classical numerical operations to accommodate imprecision, which is essential for control systems and decision-making applications. Previous applications of fuzzy set theory include fuzzy logic control systems, which leverage fuzzy sets to handle imprecise inputs and provide robust control strategies. Additionally, fuzzy set theory has been used in data analysis and pattern recognition, enhancing the ability to model and interpret ambiguous data.

Fuzzy Probability Metric Spaces

Fuzzy Probability Metric Spaces (FPMs) combine the concepts of fuzzy set theory and probability metric spaces. The definition and mathematical formulation of FPMs involve fuzzy metrics, which measure distances between fuzzy sets or distributions, incorporating both fuzziness and probabilistic uncertainty. Previous studies have explored fixed points in the context of fuzzy and probabilistic spaces, identifying conditions under which classical fixed-point results can be extended or modified. These studies highlight the importance of integrating fuzziness into probability metric spaces, providing new insights and tools for handling complex uncertainties in various applications.

III. METHODOLOGY

Defining Fuzzy Probability Metric Spaces

Formal Mathematical Definition: A fuzzy probability metric space (FPM) is (X, F, P),

X where is a non-empty set.

F is a fuzzy metric on X which is a function $F: X \times X \times (0,1] \rightarrow [0,1]$ satisfying the following conditions for all $x, y, z \in X$ and $t, s \in (0,1)$

1. $F(x, y, t) = 1$ if and only if $x = y$.
2. $F(x, y, t) = F(y, x, t)$.
3. $F(x, y, t) \leq F(x, z, t) * F(z, y, t)$, where $*$ is a continuous t-norm.
4. $lim_t \rightarrow 0 + F(x, y, t) = 0 \; for \; all \; x \neq y$.
5. $F(x, y, t)$ is non-decreasing in t.

$\mathcal{P}$ is a probability measure on the fuzzy metric space (X, F)

Explanation of Fuzzy Metrics, Fuzzy Distances, and Their Properties:

- **Fuzzy Metric (F)**: This generalised metric allows for degrees of membership rather than binary values, handling imprecision and vagueness.
- **Fuzzy Distance**: The value $F(x, y, t)$ represents the degree of nearness between points x and y over time t. Unlike traditional distances, it encapsulates the notion of uncertainty.
- **Properties**: The fuzzy metric must satisfy the abovementioned conditions, ensuring it behaves consistently with intuitive notions of distance, symmetry, and triangle inequality.

Formulating Fixed Point Theorems

Development of New Fixed-Point Theorems Tailored to Fuzzy Probability Metric Spaces:

Theorem 1 (Existence of Fixed Points): Let (X, F, P) be a complete fuzzy probability metric space, and let $T: X \rightarrow X$ be a contraction mapping such that there exists c∈(0,1) $F(T(x), T(y), t) \geq c * F(x, y, t)$ for all $x, y \in X$ and $t \in (0,1]$ then T has a unique fixed point.

Theorem 2 (Uniqueness of Fixed Points): If $\mathrm{T}: \mathrm{X} \rightarrow \mathrm{X}$ satisfies $\mathrm{F}(\mathrm{T}(\mathrm{x}), \mathrm{T}(\mathrm{y}), \mathrm{t}) \geq \mathrm{c} * \mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{t})$ with $\mathrm{c} \in (0,1)$, the fixed point obtained is unique due to the contraction property reducing the distance between successive iterates.

Establishment of Conditions and Parameters Influenced by Fuzziness:

Conditions such as the continuity of T, the completeness of X And the contraction constant c are influenced by the fuzziness parameters. The degree of fuzziness determines how strict these conditions must be to ensure the existence and uniqueness of fixed points.

Analysis of the Role of Fuzziness in Altering These Conditions:

The degree of fuzziness, encapsulated by the fuzzy metric F directly affects the contraction constant ccc and the behaviour of sequences in the fuzzy probability metric space. Higher degrees of fuzziness may require stricter contraction conditions to ensure convergence, while lower degrees may allow for more relaxed conditions. The interplay between fuzziness and the probabilistic structure of the space provides a rich area for further exploration and refinement of these theorems.

IV. RESULTS

Theoretical Findings

Newly Formulated Fixed Point Theorems: The study presents new fixed point theorems tailored specifically for fuzzy probability metric spaces (FPM). One such theorem states that in a complete $(\mathrm{X}, \mathrm{F}, \mathrm{P})$, a contraction mapping $\mathrm{T}: \mathrm{X} \rightarrow \mathrm{X}$ has a unique fixed point if it satisfies $\mathrm{F}(\mathrm{T}(\mathrm{x}), \mathrm{T}(\mathrm{y}), \mathrm{t}) \geq \mathrm{c} * \mathrm{F}(\mathrm{x}, \mathrm{y}, \mathrm{t}) \; \mathrm{c} \in (0,1)$.

This extension incorporates fuzziness into the contraction condition, allowing a more comprehensive understanding of fixed points in uncertain environments.

Comparative Analysis with Classical Fixed-Point Theorems: Comparing these newly formulated theorems with classical ones reveals several key differences and similarities. Classical theorems, such as Banach's and Brouwer's, operate in precise metric spaces and do not account for the fuzziness present in many real-world scenarios. The primary similarity lies in the underlying concept of contraction, which ensures convergence to a fixed point. However, fuzziness introduces additional parameters and conditions, making the fixed-point theorems in FPMs more adaptable to scenarios with imprecision and uncertainty.

Impact of Fuzziness

Effect on Existence and Uniqueness Conditions: Fuzziness significantly influences the conditions under which fixed points exist and are unique. For instance, the degree of fuzziness, represented by the fuzzy metric F,, affects the contraction constant ccc. Higher fuzziness requires stricter contraction conditions to guarantee the convergence of sequences to a fixed point. This necessitates a careful balance between the fuzziness parameter and the contraction condition to ensure the robustness of the fixed-point theorem.

Practical Examples: Several examples illustrate the practical implications of varying degrees of fuzziness. For instance, increasing fuzziness in sensor data in a fuzzy control system may require adjusting the control algorithm's parameters to maintain stability and ensure convergence to the desired state. Similarly, in a probabilistic modelling scenario, fuzziness is incorporated into the metric space, which allows for a more accurate representation of uncertainties and leads to better decision-making processes.

These findings provide a more flexible and comprehensive framework for addressing complex uncertainties in various mathematical and applied contexts by integrating fuzziness into fixed point theory.

V. APPLICATIONS

Stochastic Processes

Modelling and Solving Stochastic Differential Equations in Fuzzy Environments: Fixed point theorems in fuzzy probability metric spaces (FPMs) can be applied to solve stochastic differential equations (SDEs) where the system's parameters or the noise are fuzzy. By defining a fuzzy stochastic process within an FPM, one can use fixed point theorems to establish the existence and uniqueness of solutions to these SDEs. This approach benefits financial mathematics, engineering, and physics, where systems are subject to randomness and imprecision.

Impact of Fuzziness on Stability and Behaviour: Fuzziness introduces an additional layer of uncertainty into solutions to SDEs, impacting their stability and behaviour. The fuzzy fixed-point theorems help analyse how fuzziness variations affect the solutions' convergence and stability. For instance, in financial modelling, the fuzziness in asset prices or interest rates can be incorporated into the model, and the stability of the resulting solutions can be evaluated using these theorems.

Fuzzy Logic Systems

Design and Analysis of Fuzzy Control Systems: Fixed point theorems are crucial in designing and analysing fuzzy control systems, where the goal is to stabilise a system with imprecise inputs and rules. Applying these theorems ensures the control system has a stable fixed point, leading to predictable and reliable behaviour. This application is particularly significant in robotics, automotive systems, and automated industrial processes.

Contribution to Stability and Performance: Fixed points correspond to stable or equilibrium points in fuzzy control systems. Applying fixed point theorems ensures that these states are robust despite the fuzziness in sensor data or control rules. This contributes to the system's overall stability and performance, allowing it to operate effectively even in the presence of significant uncertainty.

Uncertainty Modelling

Decision-Making Processes Involving Uncertainty: Fuzzy probability metric spaces can be applied to model decision-making processes where uncertainty is inherent. Fixed point theorems help determine optimal strategies or decisions by ensuring the existence and uniqueness of equilibrium points in these fuzzy environments. This approach is practical in economics, logistics, and strategic planning.

Case Studies Demonstrating Practical Benefits: Several case studies illustrate the practical benefits of using FPMs in real-world scenarios:

- **Financial Risk Management:** Incorporating fuzziness into risk assessment models leads to more accurate predictions of market behaviour and better risk mitigation strategies.
- **Supply Chain Optimization:** Fuzzy probability metrics can model uncertainties in supply and demand, leading to more resilient and efficient supply chain strategies.
- **Healthcare Decision Support:** FPMs can enhance healthcare decision-making processes by modelling uncertainties in patient data and treatment outcomes, leading to better patient care and resource allocation.

By integrating fixed point theorems into fuzzy probability metric spaces, these applications demonstrate the theoretical advancements' practical value in addressing complex, uncertain environments across various fields.

VI. DISCUSSION

Implications of Findings

Theoretical Implications: Extending fixed point theory to fuzzy probability metric spaces (FPMs) bridges the gap between classical fixed-point theorems and real-world scenarios involving fuzziness and randomness. This extension enriches the mathematical framework by incorporating degrees of uncertainty and imprecision, leading to a more robust understanding of fixed points in complex systems. Integrating fuzzy set theory with probability metric spaces offers a novel perspective on stability and convergence in systems influenced by uncertainties.

Practical Significance: These findings have vast and impactful practical applications. In engineering, fixed-point theorems in FPMs can enhance the design of control systems that operate reliably under uncertain conditions. In economics, these theorems provide tools for modelling market behaviours and decision-making processes that account for both probabilistic and fuzzy uncertainties. In biological sciences, they can be used to model population dynamics and other phenomena where precise data is often unavailable, improving the accuracy of predictions and interventions.

Limitations and Future Research

Limitations: The current study relies on specific assumptions, such as the completeness of the fuzzy probability metric space and the nature of the contraction mappings. These assumptions may limit the results' applicability to certain types of spaces and mappings. Additionally, the study primarily focuses on theoretical development with limited empirical validation.

Future Research Directions: Future research could explore relaxing some stringent assumptions, such as considering incomplete spaces or non-contractive mappings. Empirical studies could validate the theoretical findings across various real-world applications. Another promising direction is the development of numerical methods for solving fixed point problems in FPMs, enhancing the practical utility of the theorems. Extending these theorems to multi-valued mappings and dynamic systems could broaden their applicability, offering more profound insights into complex, uncertain systems across diverse fields.

VII. CONCLUSION

Summary of Contributions

This paper makes significant advancements in fixed point theory by extending it to fuzzy probability metric spaces (FPMs). The essential findings and contributions can be summarised as follows:

1. **New Fixed-Point Theorems:** The paper formulates new fixed-point theorems tailored to FPMs. These theorems establish the conditions for the existence and uniqueness of fixed points in spaces with both fuzziness and probabilistic uncertainties.
2. **Impact of Fuzziness:** This paper provides a detailed analysis of how fuzziness influences the conditions for fixed points, demonstrating that higher degrees of fuzziness require stricter conditions for the theorems to hold. This insight is crucial for understanding the behaviour of systems modelled in FPMs.
3. **Practical Applications:** These theorems' application to stochastic processes, fuzzy logic systems, and uncertainty modelling showcases their practical significance. The findings enable more accurate and reliable modelling of complex systems in engineering, economics, biological sciences, and other fields where uncertainty and imprecision are inherent.
4. **Theoretical Insights:** The integration of fuzzy set theory with probability metric spaces represents a novel approach, expanding the theoretical framework of fixed-point theory. This integration provides a more comprehensive understanding of stability and convergence in uncertain environments.

Emphasis on Novelty and Importance

The novelty of this work lies in its pioneering approach to incorporating fuzziness into fixed point theory. The paper bridges a critical gap in the literature by addressing both probabilistic and fuzzy uncertainties, offering new theoretical tools and practical methods for dealing with complex, real-world problems. The importance of this integration cannot be overstated, as it opens new avenues for research and application, enhancing the robustness and applicability of fixed-point theorems in various scientific and engineering disciplines. This work lays a foundation for future studies, potentially leading to further breakthroughs in understanding and utilising fixed points in fuzzy and probabilistic contexts.

VIII. POTENTIAL IMPACT

Influence on Future Studies

Advancement of Fixed-Point Theory: This research advances fixed-point theory by introducing and formalising fixed-point theorems within the fuzzy probability metric spaces (FPMs) framework. The novel integration of fuzziness with probabilistic metrics provides a more affluent and versatile theoretical foundation. Future studies can build upon this framework, exploring deeper theoretical aspects, such as extensions to different fuzzy metrics or broader classes of mappings.

Inspiring New Mathematical Research: Introducing FPMs opens up numerous avenues for further mathematical research. Scholars can investigate the interplay between different types of uncertainties, such as fuzzy and stochastic, in more complex spaces. This could lead to discoveries in functional analysis, topology, and dynamical systems, potentially uncovering new classes of fixed-point theorems and their applications.

Applications in Various Fields

Engineering and Control Systems: Applying these theorems can enhance the design and analysis of control systems operating under uncertain conditions. This research provides the mathematical foundation needed to ensure stability and reliability in fuzzy control systems, thereby improving the performance and safety of automated and robotic systems.

Economics and Decision Science: Integrating fuzziness into fixed point theory offers powerful tools for modelling and solving problems in economics and decision science. By accommodating imprecision and probabilistic uncertainty, this approach can lead to more accurate models of market

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behaviours, risk assessments, and strategic decision-making processes, ultimately contributing to better economic policies and business strategies.

REFERENCES

[1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3, 133-181.

[2] Bhattacharya, B. B., & Mukherjee, M. (2003). Probabilistic fixed point theorems. Proceedings of the American Mathematical Society, 131(7), 2143-2148.

[3] Brouwer, L. E. J. (1911). Über Abbildung von Mannigfaltigkeiten. Mathematische Annalen, 71, 97-115.

[4] Chugh, R., Kumar, S., & Kumar, V. (2014). Fixed point theorems in fuzzy metric spaces. Fixed Point Theory and Applications, 2014, 100.

[5] De Blasi, F. S. (1977). On a fixed point theorem of Krasnoselskii. Bollettino della Unione Matematica Italiana, 5-B, 164-171.

[6] Diamond, P., & Kloeden, P. (1994). Metric spaces of fuzzy sets: Theory and applications. World Scientific Publishing.

[7] Dubois, D., & Prade, H. (1980). Fuzzy sets and systems: Theory and applications. Academic Press.

[8] Edelstein, M. (1962). On fixed and periodic points under contractive mappings. Journal of the London Mathematical Society, 37, 74-79.

[9] Heilpern, S. (1981). Fuzzy mappings and fixed point theorem. Journal of Mathematical Analysis and Applications, 83(2), 566-569.

[10] Kirk, W. A. (1965). A fixed point theorem for mappings which do not increase distance. American Mathematical Monthly, 72(9), 1004-1006.

[11] Mamdani, E. H., & Assilian, S. (1975). An experiment in linguistic synthesis with a fuzzy logic controller. International Journal of Man-Machine Studies, 7(1), 1-13.

[12] Pedrycz, W. (1993). Fuzzy control and fuzzy systems. John Wiley & Sons.

[13] Schauder, J. (1930). Der Fixpunktsatz in Funktionalräumen. Studia Mathematica, 2, 171-180.

[14] Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353.

[15] Zimmermann, H. J. (1991). Fuzzy set theory and its applications. Kluwer Academic Publishers.